The combinatorics of gaps between prime numbers

Fred B. Holt, with Helgi Rudd
fbholt62@gmail.com

A presentation of a paper at
Connections in Discrete Mathematics
In honor of Ron Graham's 80th birthday

Simon Fraser University
15-19 June 2015

Overview

- We study the <u>cycle of gaps</u> at each stage of Eratosthenes sieve.
  - Identify the recursion from one cycle of gaps to the next
  - Construct a discrete dynamic model
    for the populations of gaps and of constellations of gaps
  - *Beautifully lucky! – simple eigenstructure*
  - Asymptotic relative populations of gaps
  - Polignac result and Hardy & Littlewood's Conjecture B
  - Strong Polignac result on CPAP

These are actual populations in the sieve, not probabilities.

The cycle of gaps G(p#)

1 2 3 5 7 9 11 13 15 17 19 21 23 25 27 29 31 33 35 37 39 41 43 45 47 49 51 53

1 2 3 5 7 9 11 13 15 17 19 21 23 25 27 29 31 33 35 37 39 41 43 45 47 49 51 53

4 2 4 2 4 2 4 2 4 2 4 2 4 2 4 2

G(3#) = 4 2

1 2 3 5 7 9 11 13 15 17 19 21 23 25 27 29 31 33 35 37 39 41 43 45 47 49 51 53

6 4 2 4 2 4 6 2 6 4 2 4 2 4

G(5#) = 6 4 2 4 2 4 6 2

Recursive construction

R1. The next prime is $p_{k+1} = g_1 + 1$

R2. Concatenate p_{k+1} copies of $G(p_k\#)$.

R3. Close gaps after first gap and
as indicated by elementwise product $p_{k+1} * G(p_k\#)$.

Example: G(5#) = 6 4 2 4 2 4 6 2 to G(7#)

$p_{k+1} = 7$

6 4 2 4 2 4 6 2 6 4 2 4 2 4 6 2 6 4 2 4 2 4 6 2 6 4 2 4 2 4 6 2 6 4 2 4 2 4 6 2 6 4 2 4 2 4 6 2 6 4 2 4 2 4 6 2

*7 * G(5#)*

42 28 14 28 14 28 42 14

10, 2 4 2 4 6 2 6 4 2 4 **6** 6 2 6 4 2 **6** 4 6 **8** 4 2 4 2 4 **8** 6 4 **6** 2 4 6 2 6 **6** 4 2 4 6 2 6 4 2 4 2 ,**10**, 2

= G(7#)

Theorem (if $g < 2p$)

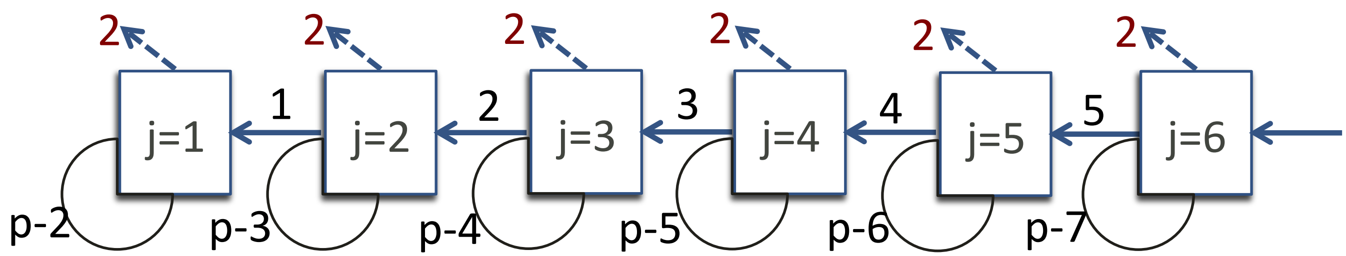
- *Lemma*: Every closure of adjacent gaps in $G(p_k\#)$ occurs exactly once in forming $G(p_{k+1}\#)$.
  - *Proof: Chinese Remainder Theorem.*

- *Theorem:* For a constellation s of length j, if $|s| < 2p_{k+1}$, then the population $n_{s,j}(p_{k+1}\#)$ is given by

$$n_{s,j}(p_{k+1}\#) = (p_{k+1}-j-1)\, n_{s,j}(p_k\#) + n_{s,j+1}(p_k\#)$$

Driving terms for s of length j+1

Discrete dynamic model for gaps



Express as a linear system:

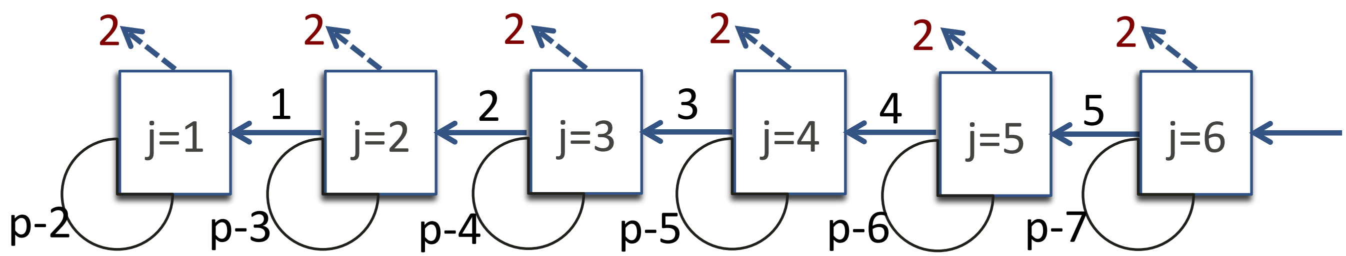
$$n_g(p_k\#) = M(p_k)\, n_g(p_{k-1}\#)$$

$$= \underbrace{M(p_k)\, M(p_{k-1}) \ldots M(p_1)}\, n_g(p_0\#)$$

$$= M^k\, n_g(p_0\#)$$

Notes: Definition of M^k.
Need p_0 with $g < 2p_1$ & initial conditions $n_g(p_0\#)$.
The system depends on p.
The initial conditions depend on g.

Transfer matrix for $n_g(p_o\#)$



$$M(p) = \begin{bmatrix} p-2 & 1 & & & & \\ & p-3 & 2 & & & \\ & & p-4 & 3 & & \\ & & & p-5 & 4 & \\ & & & & \ddots & \ddots \end{bmatrix}$$

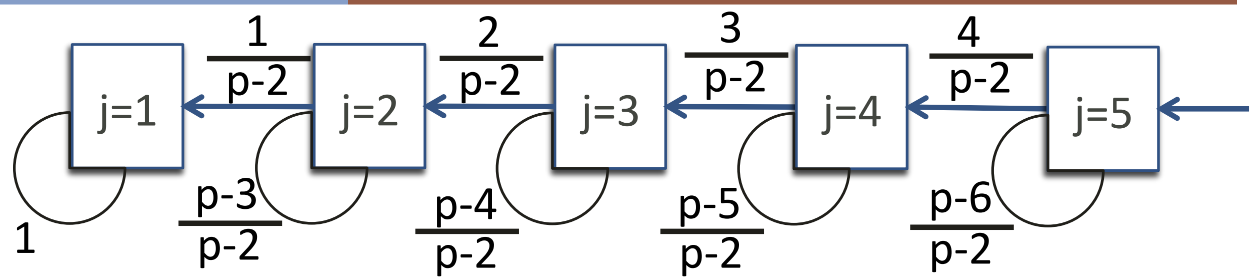
Populations of gaps grow super-exponentially by factors of (p_k-2).

So we normalize by the population of 2's,
which has no driving terms with j > 1.

$$n_{2,1}(p\#) = \prod_{3}^{p} (q-2)$$

Normalized model for gaps

$$w_{g,j}(p\#) = \frac{n_{g,j}(p\#)}{n_{2,1}(p\#)}$$



Discrete model for relative populations of a gap g to the gap 2:

$$w_g(p_k\#) = M(p_k)\, w_g(p_{k-1}\#) = M^k\, w_g(p_0\#)$$

with transfer matrix

$$M(p) = \begin{bmatrix} 1 & \frac{1}{p-2} & & & \\ & \frac{p-3}{p-2} & \frac{2}{p-2} & & \\ & & \frac{p-4}{p-2} & \frac{3}{p-2} & \\ & & & \ddots & \ddots \end{bmatrix}$$

Eigenstructure – beautiful!

$$
M(p) = \begin{bmatrix} 1 & \frac{1}{p-2} & & & \\ & \frac{p-3}{p-2} & \frac{2}{p-2} & & \\ & & \frac{p-4}{p-2} & \frac{3}{p-2} & \\ & & & \ddots & \ddots \end{bmatrix}
$$

$$
= \underbrace{\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & \\ & 1 & -2 & 3 & -4 & \\ & & 1 & -3 & 6 & \ddots \\ & & & 1 & -4 & \\ & & & & 1 & \\ & & & & & \ddots \end{bmatrix}}_{R}
\underbrace{\begin{bmatrix} 1 & & & \\ & \frac{p-3}{p-2} & & \\ & & \frac{p-4}{p-2} & \\ & & & \ddots \end{bmatrix}}_{\Lambda}
\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \\ & 1 & 2 & 3 & 4 & \\ & & 1 & 3 & 6 & \ddots \\ & & & 1 & 4 & \\ & & & & 1 & \\ & & & & & \ddots \end{bmatrix}}_{L}
$$

Simple Eigenstructure for M^k

$$M^k = M(p_k)\, M(p_{k-1})\, M(p_{k-2}) \ldots M(p_1)$$

$$= \quad R \quad \Lambda^k \quad L$$

$$= \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & \\ & 1 & -2 & 3 & -4 & \\ & & 1 & -3 & 6 & \ddots \\ & & & 1 & -4 & \\ & & & & 1 & \\ & & & & & \ddots \end{bmatrix} \begin{bmatrix} 1 & & \\ & \prod_{p_1}^{p_k} \frac{p-3}{p-2} & \\ & & \prod_{p_1}^{p_k} \frac{p-4}{p-2} \\ & & & \ddots \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \\ & 1 & 2 & 3 & 4 & \\ & & 1 & 3 & 6 & \ddots \\ & & & 1 & 4 & \\ & & & & 1 & \\ & & & & & \ddots \end{bmatrix}$$

Asymptotics

Expand the right-hand side, in terms of the eigenstructure:

$$w_g(p_k\#) = M^k\, w_g(p_0\#)$$

$$= (L_1\, w_g(p_0\#))\, R_1 \;+\; \lambda_2^k\, (L_2\, w_g(p_0\#))\, R_2 \;+\; \lambda_3^k\, (L_3 w_g(p_0\#))\, R_3 + \ldots$$

We have the very simple values: $L_1 = \langle 1 \rangle$, $\lambda_1 = 1$, $R_1 = e_1$

So the asymptotic ratio of the population of the gap g to the gap 2 is given by

$$w_g^{\infty} = \sum_j w_{g,j}(p_0\#)$$

Condition: p_0 is such that $g < 2p_1$

Polignac result and HL Conjecture B

- *Polignac conjecture*: every even g occurs as a gap between consecutive primes infinitely often

- *Hardy & Littlewood Conjecture B*: for any even g, the number of prime pairs p and p+g such that p+g < n is approximately

$$2C_2 \frac{n}{(\log n)^2} \prod_{\substack{q > 2 \\ q \mid g}} \frac{q-1}{q-2}$$

Our result: for every even g,
the gap g arises in Eratosthenes sieve
and its relative population tends toward

$$w_g^{\infty} = \prod_{\substack{q > 2 \\ q \mid g}} \frac{q-1}{q-2}$$

Examples

For p_0=5,   G(5#)= 6 4 2 4 2 4 6 2

	g=2	**4**	**6**	**8**	**10**	**12**	**14**
j=1	3	3	2	0	0	0	0
j=2	0	0	4	2	2	0	0
j=3	0	0	0	1	2	4	1
j=4	0	0	0	0	0	2	2
$\sum n_g$	**3**	**3**	**6**	**3**	**4**	**6**	**3**
$\sum w_g$	**1**	**1**	**2**	**1**	**4/3**	**2**	**1**
w_g^{∞}	**1**	**1**	**2**	**1**	**4/3**	**2**	**6/5**

Examples

For p_0=5, G(5#)= 6 4 2 4 2 4 6 2

	g=2	**4**	**6**	**8**	**10**	**12**	**14**
j=1	3	3	2	0	0	0	0
j=2	0	0	4	2	2	0	0
j=3	0	0	0	1	2	4	1
j=4	0	0	0	0	0	2	2
$\sum n_g$	**3**	**3**	**6**	**3**	**4**	**6**	**3**
$\sum w_g$	**1**	**1**	**2**	**1**	**4/3**	**2**	**1**
w_g^∞	**1**	**1**	**2**	**1**	**4/3**	**2**	**6/5**

Outline of proof: cycle of gaps G(n)

For even number g,
let Q be the product of unique prime factors dividing g.

Form G(Q).

Recursion from G(n) to G(pn).

Each of the $\phi(Q)$ gaps starts a driving term for g
– by going g/Q times around the cycle G(Q).

Now fill in G(Q) to obtain G(p#).

Proof (cont'd): asymptotics

Let q be the largest prime factor of Q.

Then $$\sum_j n_{g,j}(q\#) = \phi(Q) \prod_{(p,Q)=1} p-2$$

We don't know the lengths, but L_1 = <1>.

and $$\sum_j w_{g,j}(q\#) = \phi(Q) \prod_{(p,Q)=1} p-2 \Big/ \prod_{p=3}^{q} p-2$$

$$= \prod_{\substack{p>2 \\ p \mid Q}} \frac{p-1}{p-2}$$

This is w_g^{∞}.

Model for constellations of length j

$$M_j(p) = \begin{bmatrix} 1 & \frac{1}{p-j-1} & & & \\ & \frac{p-j-2}{p-j-1} & \frac{2}{p-j-1} & & \\ & & \frac{p-j-3}{p-j-1} & \frac{3}{p-j-1} & \\ & & & \ddots & \ddots \end{bmatrix}$$

Normalize by p-j-1.

$$M_j^k = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & \\ & 1 & -2 & 3 & -4 & \\ & & 1 & -3 & 6 & \ddots \\ & & & 1 & -4 & \\ & & & & 1 & \ddots \end{bmatrix} \begin{bmatrix} 1 & & \\ & \prod_{p_1}^{p_k} \frac{p-j-2}{p-j-1} & \\ & & \prod_{p_1}^{p_k} \frac{p-j-3}{p-j-1} \\ & & & \ddots \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \\ & 1 & 2 & 3 & 4 & \\ & & 1 & 3 & 6 & \ddots \\ & & & 1 & 4 & \\ & & & & 1 & \ddots \end{bmatrix}$$

$$R \qquad \Lambda^k \qquad L$$

Strong Polignac result on CPAP

- j+1 consecutive primes in arithmetic progression correspond to a constellation gg…g of length j.
- *Feasibility*: Let gg..g be a constellation of length j, then g must be divisible by every prime q <= j+1.

Our result: Let g be any even number, then for *every feasible j* the constellation gg…g arises in Eratosthenes sieve with relative population approaching

$$w^{\infty}_{gg..g} = \frac{\prod_{q>2;\, q \mid g} q-1}{\prod_{q>j+1;\, q \mid g} q-j-1}$$

Examples

gg..g	j	Q	w^{∞}
6 6	2	6	2
6 6 6	3	6	2
12, 12, 12	3	6	2
72, 72, 72	3	6	2
30, 30, 30	3	30	8
396, 396, 396	3	66	20/7
30, 30, 30, 30	4	30	8
3060, 3060, 3060, 3060	4	510	32/3
30, 30, 30, 30, 30	5	30	8

On twin primes (g=2)

Constellations with 2's:

s	**j**	$\mathbf{w^{\infty}}$
2 4 2	3	1
2 10 2	3	8/3
2 10 2 10 2	5	144/35
2 10 2 10 2 4 2 10 2 10 2	11	24

six pairs of twin primes in a span of 56 - !!

Surviving the sieve

How does survival work, from the cycle of gaps G(p#) to gaps between primes?

How does the model for survival depend on the length j?

Is survival at all fair?