Patterns among the Primes

Eratosthenes sieve as a discrete dynamic system

Joint Mathematics Meetings 2024
San Francisco

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https://www.primegaps.info

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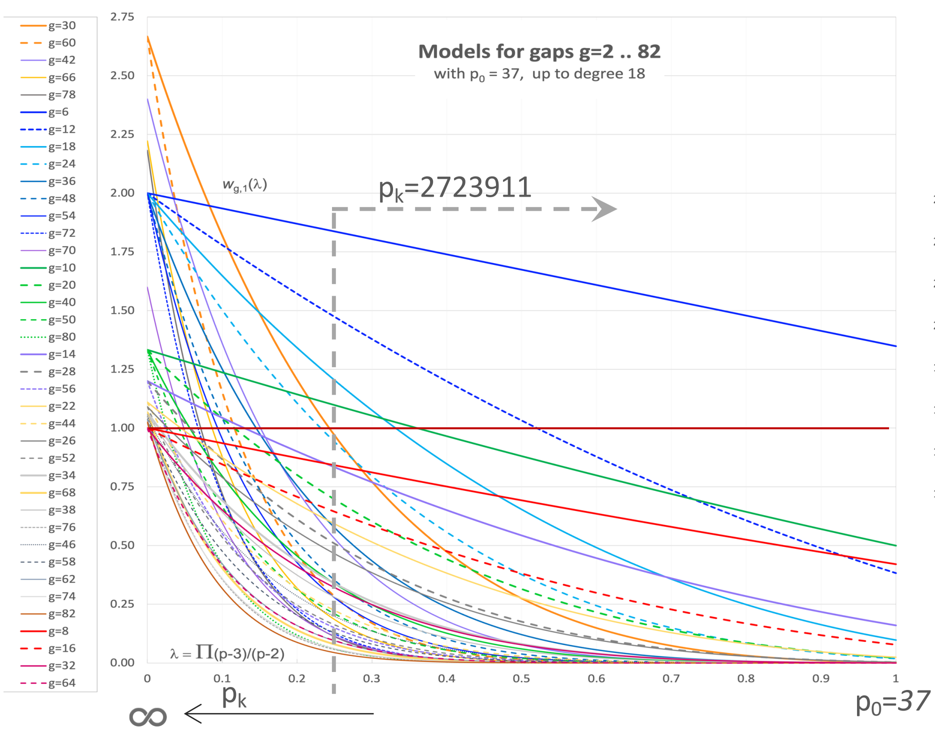
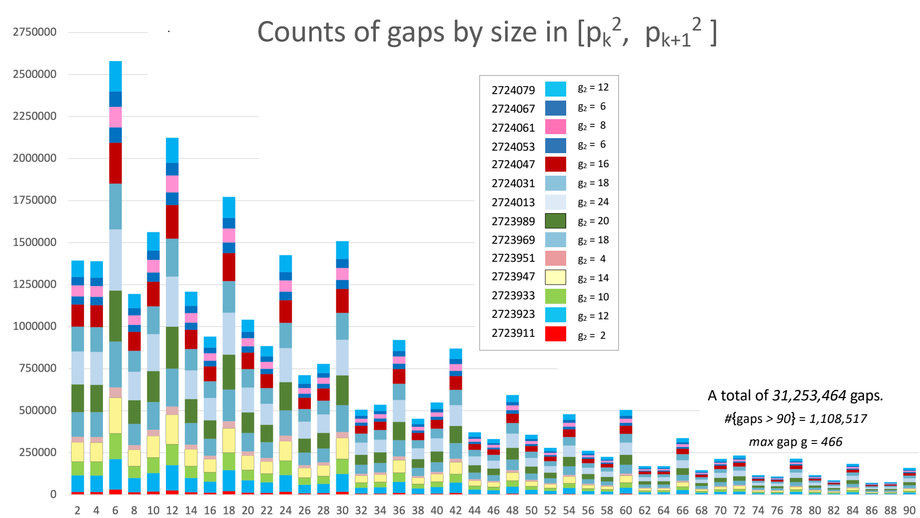
We study Eratosthenes sieve as a discrete dynamic system.

We develop <u>exact</u> models for the relative populations of the gaps among the candidate prime numbers, across stages of the sieve.



Samples of the gaps between primes agree with these models to first order.



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Eratosthenes Sieve

List the natural numbers (up to some bound *N*).

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 …

For *p=2:*

1 2 3 5 7 9 11 13 15 17 19 21 23 25 27 29 31 33 …

2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2

For *p=3.*

1 2 3 5 7 11 13 17 19 23 25 29 31 …

Unit

Confirmed primes

Remaining candidate primes or "3-rough" numbers

1 5 7 11 13 17 19 23 25 29 31 …

4 2 4 2 4 2 4 2 4 2 4

For *p=5.*

1 7 11 13 17 19 23 29 31 …

6 4 2 4 2 4 6 2 6

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Cycles of gaps $\mathcal{G}(p\#)$

At each stage of Eratosthenes sieve, there is a corresponding *cycle of gaps* $\mathcal{G}(p\#)$ among the remaining candidate primes.

$$\mathcal{G}(2\#) = 2$$

$$\mathcal{G}(3\#) = 4\ 2$$

$$\mathcal{G}(5\#) = 6\ 4\ 2\ 4\ 2\ 4\ 6\ 2$$

The ***length*** of this cycle $\mathcal{G}(p\#)$ is $\phi(p\#)$ gaps and its ***span*** (the sum of its gaps) is *p#*.

> Notation ***p#*** denotes the *primorial of p.* This is the product of all the prime numbers up to and including *p.*
> *2# = 2   3# = 6   5# = 30*

The first gap in the cycle goes from the unit 1 to the next prime, passing over the confirmed prime numbers.

Recursion across the cycles of gaps $\mathcal{G}(p\#)$

There is a 3-step recursion that produces the next *cycle of gaps* from the current one.

$$\mathcal{G}(p_k\#) \longrightarrow \mathcal{G}(p_{k+1}\#)$$

Recursion. For p_k we have the cycle of gaps $\mathcal{G}(p_k\#) = g_1\, g_2\, g_3 \ldots g_N$

R1. The next prime $p_{k+1} = g_1+1$

R2. Concatenate p_{k+1} copies of $\mathcal{G}(p_k\#)$

R3. Fusions: Add together the gaps g_1+g_2 and thereafter add adjacent gaps at the running sums indicated by the elementwise product $p_{k+1} * \mathcal{G}(p_k\#)$

This is the dynamic system.

$$\mathcal{G}(p_0\#) \longrightarrow \mathcal{G}(p_1\#) \longrightarrow \ldots \longrightarrow \mathcal{G}(p_k\#) \longrightarrow \mathcal{G}(p_{k+1}\#) \longrightarrow \mathcal{G}(p_{k+2}\#) \longrightarrow$$

Initial conditions

Example: $\mathcal{G}(5\#) \longrightarrow \mathcal{G}(7\#)$.

p_k=5 and $\mathcal{G}(5\#)$ = 6 4 2 4 2 4 6 2

R1. Next prime: p_{k+1} = 6+1 = 7

R2. Concatenate 7 copies of $\mathcal{G}(5\#)$

6 4 2 4 2 4 6 2 6 4 2 4 2 4 6 2 6 4 2 4 2 4 6 2 6 4 2 4 2 4 6 2 6 4 2 4 2 4 6 2 6 4 2 4 2 4 6 2 6 4 2 4 2 4 6 2

R3. Fusions

7$\mathcal{G}(5\#)$ = 42 28 14 28 14 28 42 14*

Elementwise product for running sums between fusions

$\mathcal{G}(7\#)$= **10** 2 4 2 4 6 2 6 4 2 4 **6** 6 2 6 4 2 **6** 4 6 **8** 4 2 4 2 4 **8** 6 4 **6** 2 4 6 2 6 **6** 4 2 4 6 2 6 4 2 4 2 **10** 2

Note – the last running sum 14 wraps around the end of the cycle, back to the first fusion.

The cycle of gaps $\mathcal{G}(7\#)$ has length 48 gaps and span 210.

Two observations

We make two very useful observations about the recursive construction of $\mathcal{G}(p_{k+1}\#)$

<u>Observation 1</u>.

In step R3 the minimum distance between fusions is $\boldsymbol{2p_{k+1}}$.

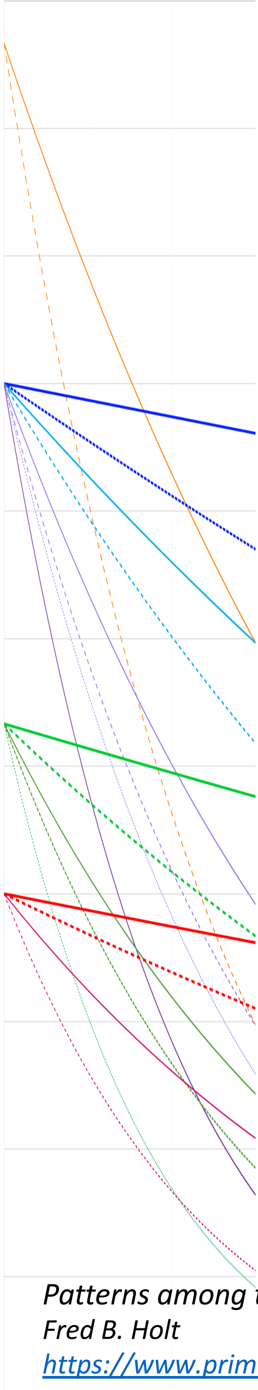
The fusions are separated by the running sums $p_{k+1} * \mathcal{G}(p_k)$, and the smallest gap in $\mathcal{G}(p_k)$ is $g=2$. So the smallest running sum is $2p_{k+1}$.

<u>Observation 2.</u>

In step R3 of $\mathcal{G}(p_k\#) \longrightarrow \mathcal{G}(p_{k+1}\#)$, each possible fusion in $\mathcal{G}(p_k\#)$ occurs exactly once.

This is a result of the Chinese Remainder Theorem.

Exercise: confirm these for $\mathcal{G}(5\#) \longrightarrow \mathcal{G}(7\#)$



Exact population models for gaps $g < 2p_1$

Observation 1

Exact Populations in $\mathcal{G}(p\#)$

$g=2$ $\quad n_2(p_{k+1}\#) = (p_{k+1}-2)\, n_2(p_k\#) \quad = \Pi_{3 \le q \le p}\,(q-2)$

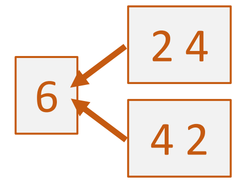
2

$g=4$ $\quad n_4(p_{k+1}\#) = (p_{k+1}-2)\, n_4(p_k\#) \quad = \Pi_{3 \le q \le p}\,(q-2)$

4

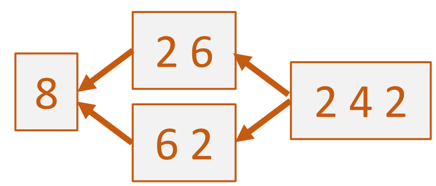
$g=6$

$$\begin{bmatrix} n_{6,1} \\ n_{6,2} \end{bmatrix} (p_{k+1}\#) = \begin{bmatrix} p_{k+1}-2 & 1 \\ 0 & p_{k+1}-3 \end{bmatrix} \begin{bmatrix} n_{6,1} \\ n_{6,2} \end{bmatrix} (p_k\#)$$

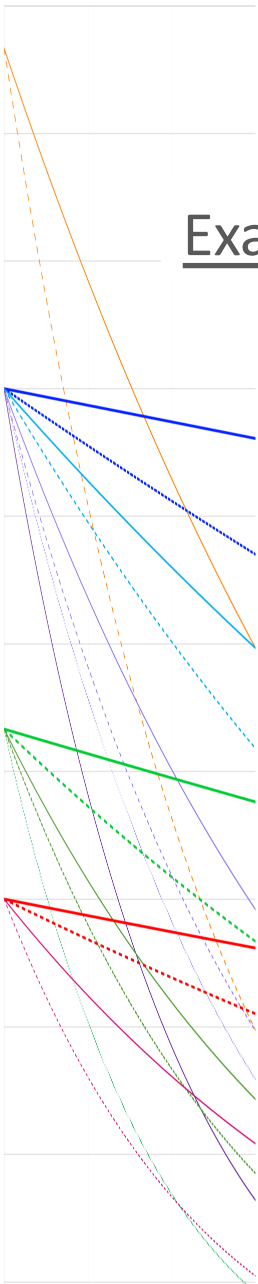


$g=8$

$$\begin{bmatrix} n_{8,1} \\ n_{8,2} \\ n_{8,3} \end{bmatrix} (p_{k+1}\#) = \begin{bmatrix} p_{k+1}-2 & 1 & 0 \\ 0 & p_{k+1}-3 & 2 \\ 0 & 0 & p_{k+1}-4 \end{bmatrix} \begin{bmatrix} n_{8,1} \\ n_{8,2} \\ n_{8,3} \end{bmatrix} (p_k\#)$$



These exact population models $n_{g,1}(p_k\#)$ are all **superexponential**, dominated by factors of $(p-2)$.

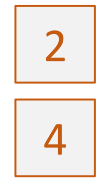


We normalize the models, dividing by $(p_{k+1}-2)$.

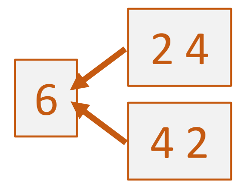
Exact models of the **relative** populations $w_{g,1}(p_k\#)$ for all $g < 2p_1$

$w_2(p_{k+1}\#) = w_2(p_k\#) = 1$

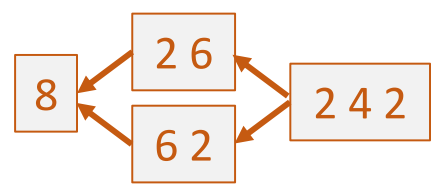
$w_4(p_{k+1}\#) = w_4(p_k\#) = 1$

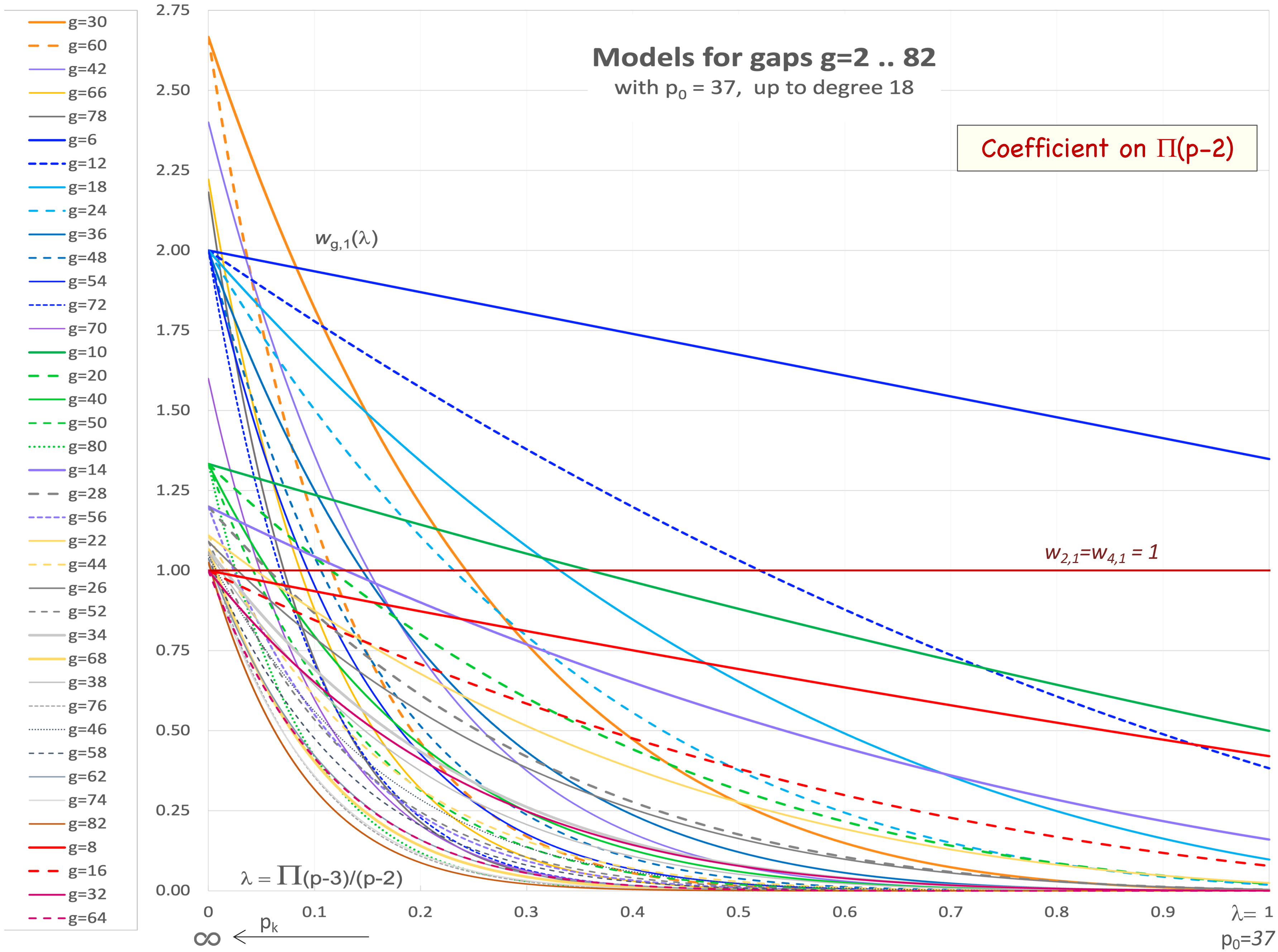


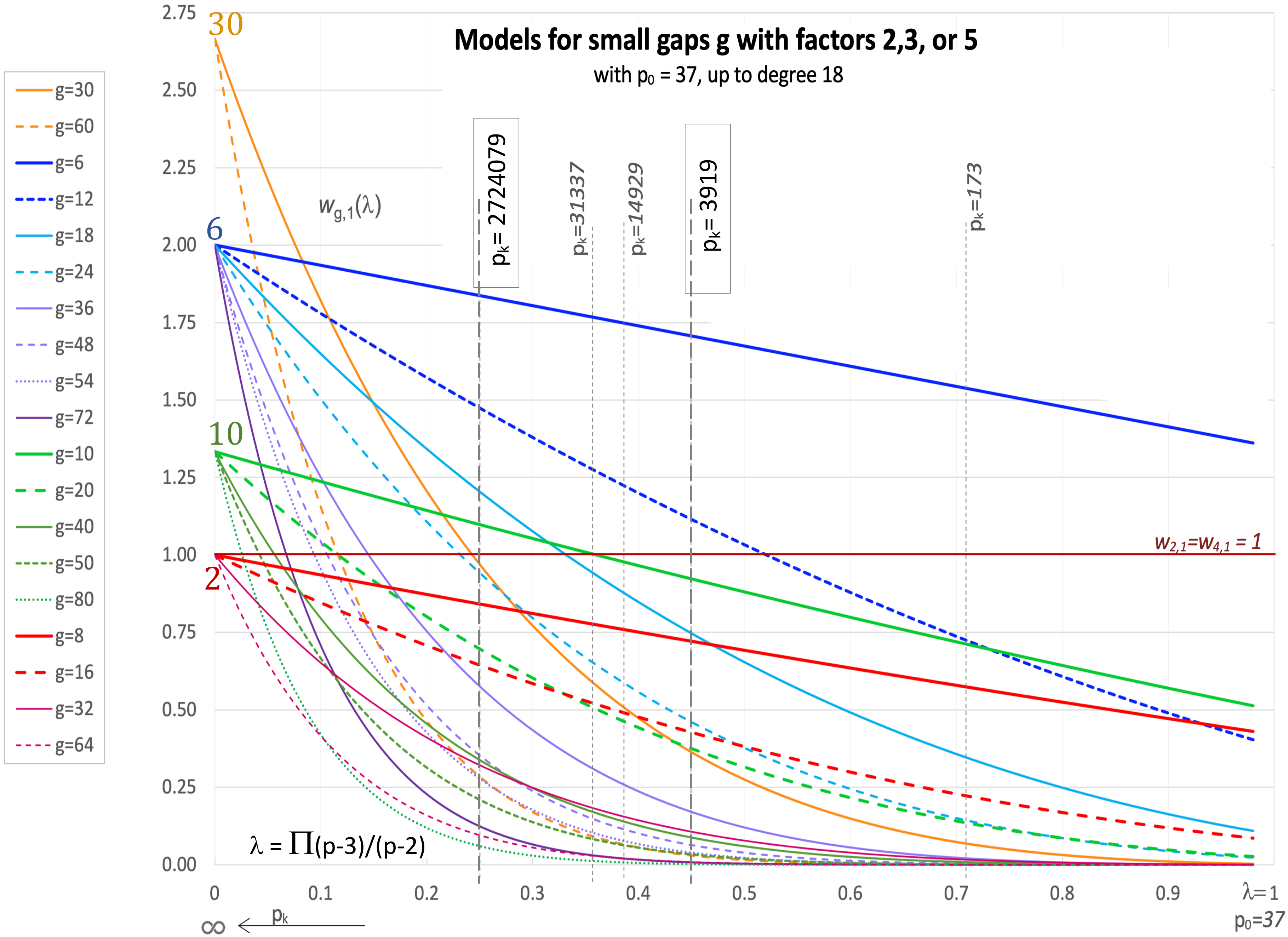
$$\begin{bmatrix} w_{6,1} \\ w_{6,2} \end{bmatrix}(p_{k+1}\#) = \begin{bmatrix} 1 & \frac{1}{p_{k+1}-2} \\ 0 & \frac{p_{k+1}-3}{p_{k+1}-2} \end{bmatrix} \begin{bmatrix} w_{6,1} \\ w_{6,2} \end{bmatrix}(p_k\#)$$



$$\begin{bmatrix} w_{8,1} \\ w_{8,2} \\ w_{8,3} \end{bmatrix}(p_{k+1}\#) = \begin{bmatrix} 1 & \frac{1}{p_{k+1}-2} & 0 \\ 0 & \frac{p_{k+1}-3}{p_{k+1}-2} & \frac{2}{p_{k+1}-2} \\ 0 & 0 & \frac{p_{k+1}-4}{p_{k+1}-2} \end{bmatrix} \begin{bmatrix} w_{8,1} \\ w_{8,2} \\ w_{8,3} \end{bmatrix}(p_k\#)$$

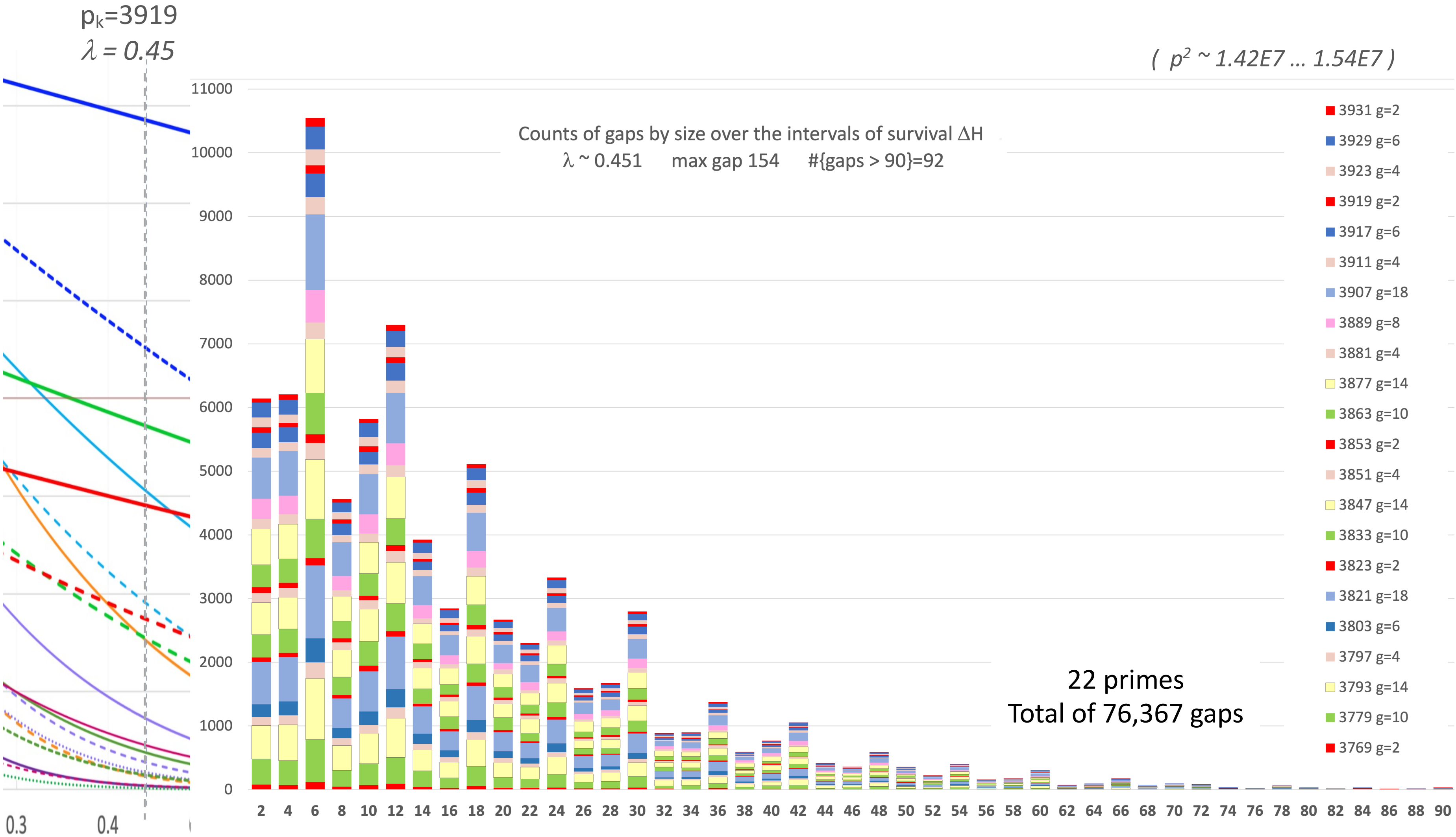



Models for gaps g=2 .. 82
with p0 = 37, up to degree 18
Coefficient on Π(p-2)
wg,1(λ)
w2,1=w4,1 = 1
λ = Π(p-3)/(p-2)
λ= 1
p0=37
pk
∞
g=30
g=60
g=42
g=66
g=78
g=6
g=12
g=18
g=24
g=36
g=48
g=54
g=72
g=70
g=10
g=20
g=40
g=50
g=80
g=14
g=28
g=56
g=22
g=44
g=26
g=52
g=34
g=68
g=38
g=76
g=46
g=58
g=62
g=74
g=82
g=8
g=16
g=32
g=64
2.75
2.50
2.25
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9



Models for small gaps g with factors 2,3, or 5
with p0 = 37, up to degree 18
g=30
g=60
g=6
g=12
g=18
g=24
g=36
g=48
g=54
g=72
g=10
g=20
g=40
g=50
g=80
g=8
g=16
g=32
g=64
wg,1(λ)
pk= 2724079
pk=31337
pk=14929
pk= 3919
pk=173
w2,1=w4,1 = 1
30
6
10
2
λ = Π(p-3)/(p-2)
2.75
2.50
2.25
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
λ=1
p0=37
∞
pk

Sampling gaps in the *Intervals of survival* $[p_k^2, p_{k+1}^2]$



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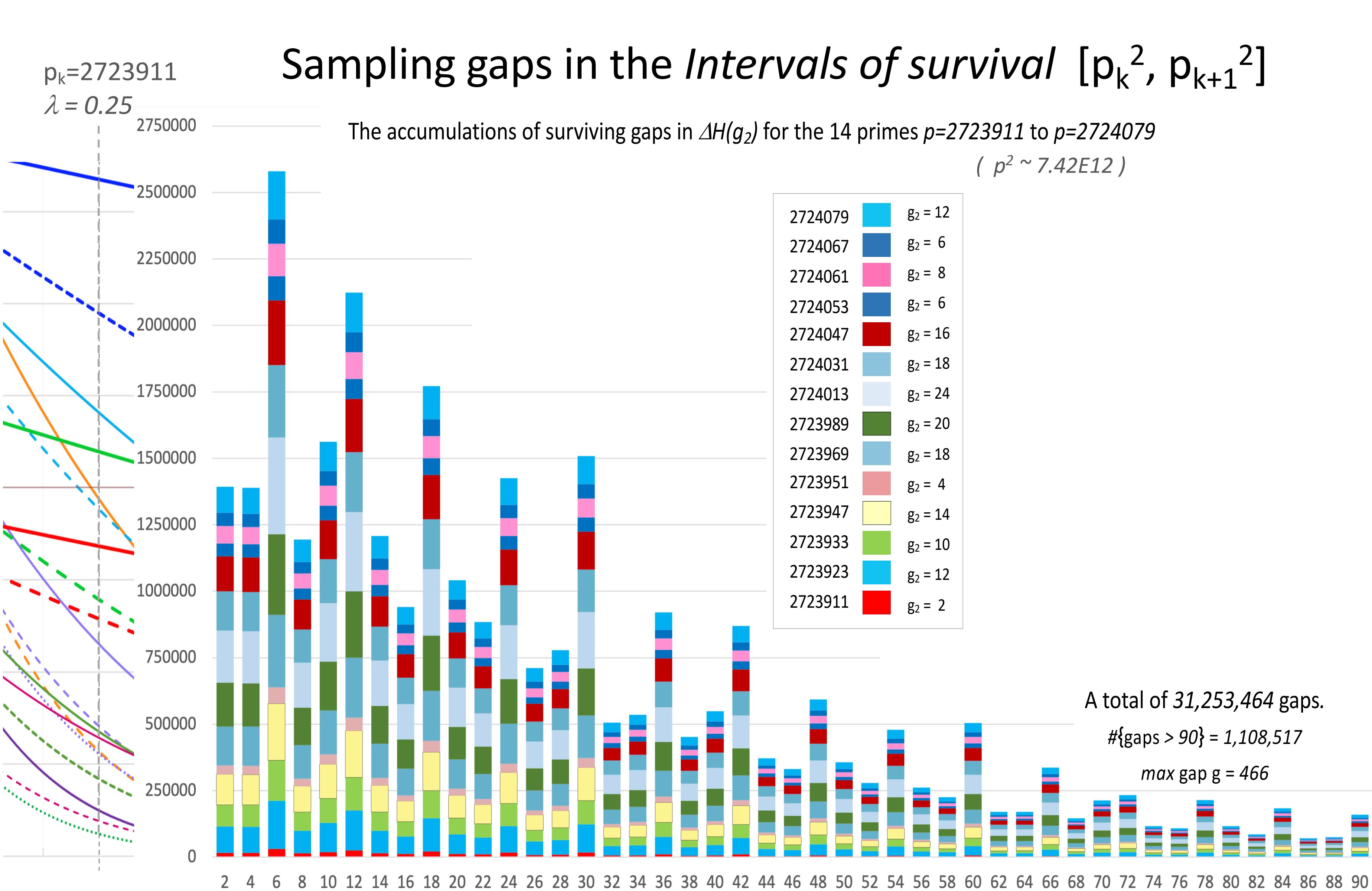
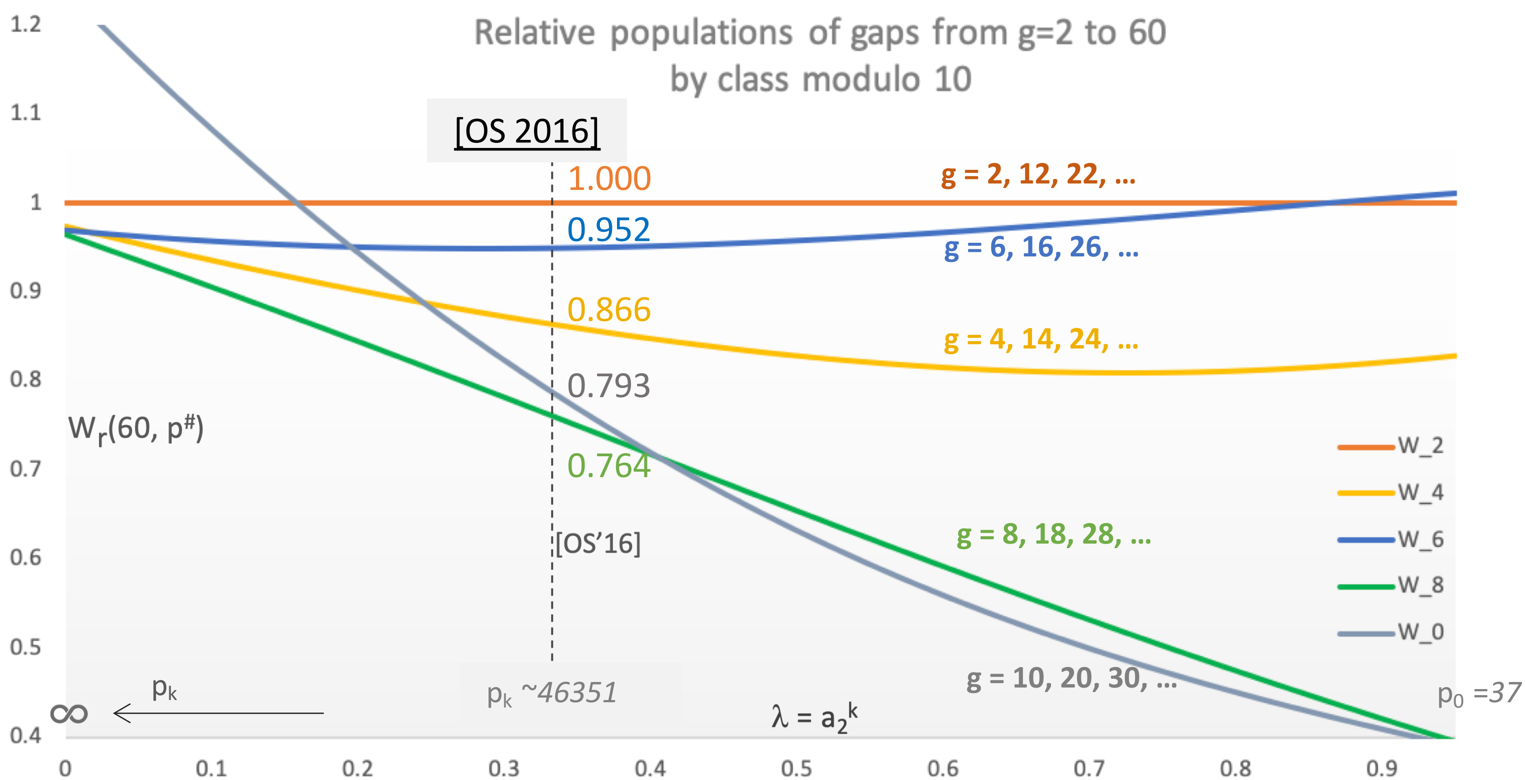


Last digits of consecutive primes (Oliver & Soundararajan 2016)

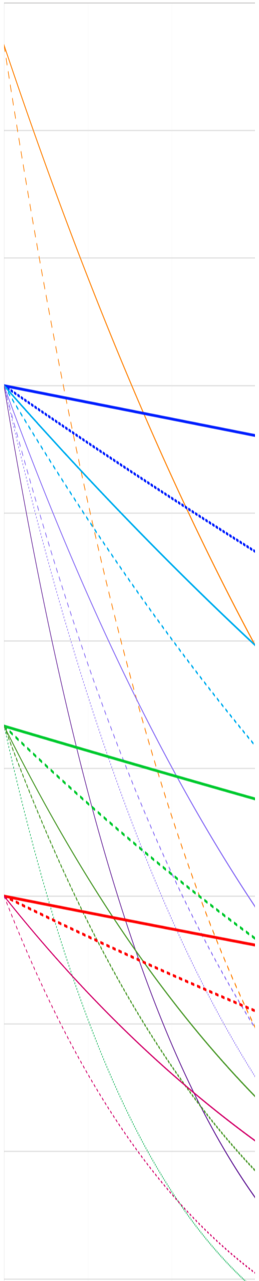
OS sampled the first 100M primes & detected biases among the gaps mod 10.

These biases are **totally expected**.



Lesson learned: computational samples do not reflect asymptotic behaviors.

The evolutions across residue classes are biased for a long time.



Results:

a. **Exact models.** Given initial conditions from $\mathcal{G}(p_0\#)$, <u>exact</u> models for relative populations of all gaps $g < 2p_1$

b. **Polignac results**

i. Every even gap g arises and persists in $\mathcal{G}(p_k\#)$

ii. Every admissible repetition, e.g. 30,30,30,30, arises and persists in $\mathcal{G}(p_k\#)$ - consecutive primes in arithmetic progression

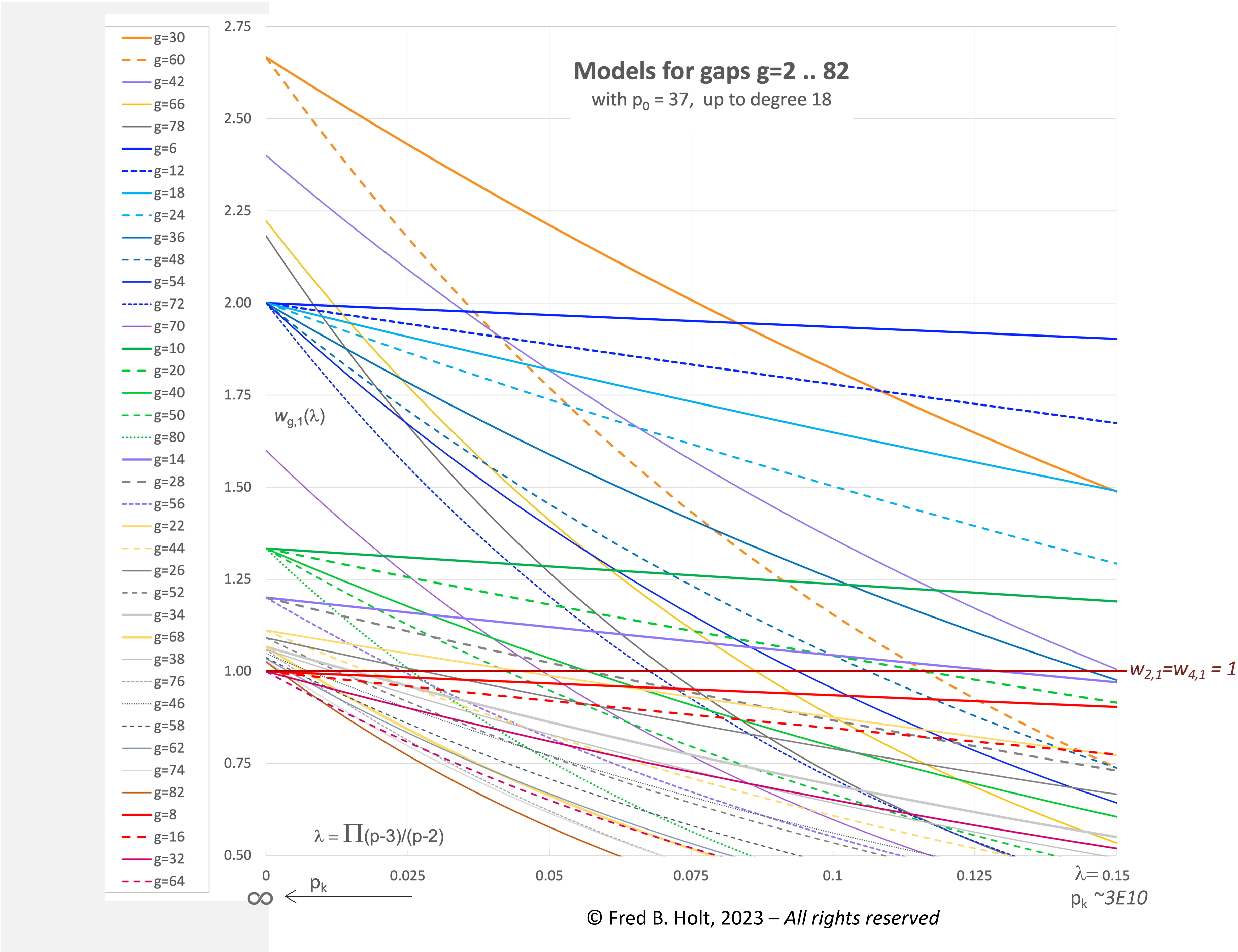
iii. The asymptotic relative population for g aligns with Hardy & Littlewood's 1923 estimates.

$$w_{g,1}(\infty) = \prod_{\text{odd } q \,|\, g} \frac{q-1}{q-2}$$

c. ***k*-tuple result.** Every admissible constellation arises and persists in $\mathcal{G}(p_k\#)$.

d. Samples confirm uniform survival, to first order…

Thank you!



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with p0 = 37, up to degree 18
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g=60
g=42
g=66
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g=24
g=36
g=48
g=54
g=72
g=70
g=10
g=20
g=40
g=50
g=80
g=14
g=28
g=56
g=22
g=44
g=26
g=52
g=34
g=68
g=38
g=76
g=46
g=58
g=62
g=74
g=82
g=8
g=16
g=32
g=64
2.75
2.50
2.25
2.00
1.75
1.50
1.25
1.00
0.75
0.50
wg,1(λ)
w2,1=w4,1 = 1
λ = Π(p-3)/(p-2)
0
0.025
0.05
0.075
0.1
0.125
λ= 0.15
pk
∞
pk ~3E10